Numerics for harmonic 1-forms on real loci of Calabi-Yau manifolds

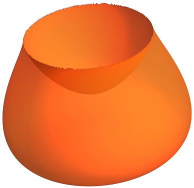
Daniel Platt (Imperial College London)
BIRS–CMI, Chennai, 28 Jan 2026

Abstract: On Calabi-Yau manifolds, there is typically no explicit formula for their Ricci-flat metrics. This often poses a problem in maths and string theory, where one would like to compute geometric data that depend on this metric. One question in this area is: is there a Calabi-Yau 3-fold so that it's real locus admits a harmonic 1-form? Such 1-forms would give rise to new G2-manifolds, but so far, no example has been proven to exist. In the talk, I will explain one new conjectural example that was obtained through numerics. This is joint work with Michael R. Douglas, Yidi Qi, and Rodrigo Barbosa. Time permitting, I will comment on the following ongoing work: many impressive calculations have been carried out with respect to numerical Calabi-Yau metrics. One interesting question is: can the results of these calculations be trusted to be near the exact results with respect to the exact Calabi-Yau metric? Answering this question for our numerical harmonic 1-forms would be interesting for a rigorous mathematical proof.

Background: Harmonic 1-forms on Calabi-Yaus

- Let Y be Calabi-Yau 3-fold with Calabi-Yau metric g_{CY}
- $\sigma : Y \to Y$ anti-holomorphic involution, $L := fix(\sigma)$
  example: quintic with real coefficients in $\mathbb{CP}^4$ and $\sigma([z_0 : \cdots : z_4]) = [\overline{z_0} : \cdots : \overline{z_4}]$
- $S^1 \times Y$ has dimension 7 and Ricci-flat. Problem: want "irreducible"
- Define $\hat{\sigma} : S^1 \times Y \to S^1 \times Y$ as $(x, y) \mapsto (-x, \sigma(y))$

$(S^1 \times Y)/\langle\hat{\sigma}\rangle$



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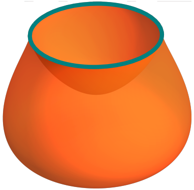

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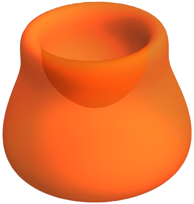

Theorem ([JK17])

If there exists $\lambda \in \Omega^1(L)$ harmonic w.r.t. $g_{CY}|_L$ that is nowhere 0, then there exists a resolution $N^7 \to (S^1 \times Y)/\langle\hat{\sigma}\rangle$ which is Ricci-flat and irreducible.

- Goal: check if such a 1-form exists
- First Betti number $\to$ harmonic 1-forms. Nowhere 0? Must know the metric!

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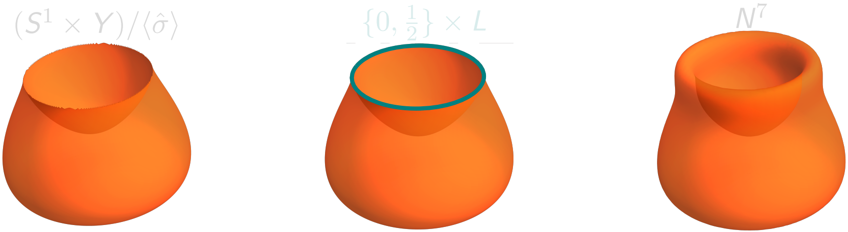

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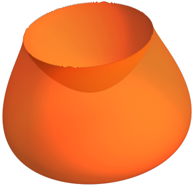

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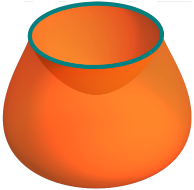

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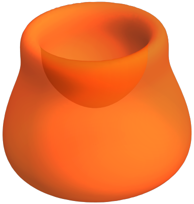

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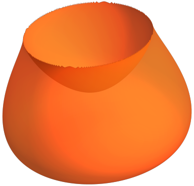

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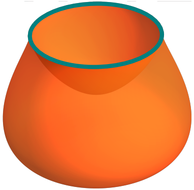

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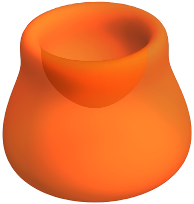

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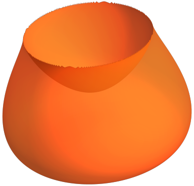

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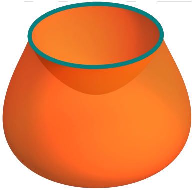

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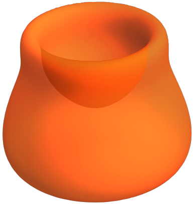

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Numerical Calabi-Yau metrics

- $Y \subset \mathbb{CP}^4$ quintic, $\Omega \in \Omega^{3,0}(Y)$, $\mathrm{vol}_\Omega := \Omega \wedge \bar{\Omega} \in \Omega^6(Y)$
- Homogeneous deg k polys $= \mathrm{span}\{s_1, \ldots, s_N\}$
- $h \in \mathbb{C}^{N \times N}$, Kähler potential $K = \log \sum_{i,j} h_{ij} s_i \overline{s_j}$, $\omega_h := i\partial\bar{\partial}K$, $\omega_h^3 = \mathrm{vol}_h \in \Omega^6(Y)$
- If $\frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} = 1$, then Ricci-flat
- [Don09]: choose $h(k)$ s.t. $\frac{\mathrm{vol}_{h(k)}}{\mathrm{vol}_\Omega} \to 1$ as $k \to \infty$
- [DLQ22, HN13, DPQB25]: choose $x_1, \ldots, x_{1000} \in Y$, find

$$\min_{h \in \mathbb{C}^{N \times N}} \int_{x_1, \ldots, x_{1000}} \left(\frac{\mathrm{vol}_h}{\mathrm{vol}_\Omega} - 1 \right)^2$$

  Large k: can take ansatz $K =$ neural network with activation $t \mapsto t^2$
- $\rightsquigarrow \omega_h$ "approximate Calabi-Yau metric"
- Alternative approaches: [HW05, JPM22, AGG$^+$21, LLRS22]

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$$\lambda = \sum_{\substack{1 \le i \le N \\ 1 \le j \le 10}} \alpha_{ij} \frac{p_i}{|x|^{d_i}} \cdot \xi_j|_L \in \Omega^1(L) \qquad \text{for} \qquad \alpha_{ij} \in \mathbb{R}$$

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$$\min_{\alpha_{ij}} \left(\int_L |\Delta \lambda| \right) / \, ||\lambda||_{L^2}$$

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Non-example 1: Fermat Quintic

- $Y := \{z = [z_0 : \cdots : z_4] \in \mathbb{CP}^4 : z_0^5 + \cdots + z_4^5 = 0\}$
- $\sigma([z_0 : \cdots : z_4]) = [\overline{z_0} : \cdots : \overline{z_4}]$
- $$\mathbb{RP}^3 \xrightarrow{\cong} L = \mathrm{fix}(\sigma) = \{x = [x_0 : \cdots : x_4] \in \mathbb{RP}^4 : x_0^5 + \cdots + x_4^5 = 0\}$$
  $$[x_0 : \cdots : x_3] \mapsto \left[x_0 : \cdots : x_3 : -\sqrt[5]{x_0^5 + \cdots + x_3^5}\right]$$
- $b^1(\mathbb{RP}^3) = 0 \Rightarrow$ no harmonic 1-form on L

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- $\sigma([z_0 : \cdots : z_4]) = [\overline{z_0} : \cdots : \overline{z_4}]$
-

$$\mathbb{RP}^3 \overset{\cong}{\to} L = \mathrm{fix}(\sigma) = \{x = [x_0 : \cdots : x_4] \in \mathbb{RP}^4 : x_0^5 + \cdots + x_4^5 = 0\}$$

$$[x_0 : \cdots : x_3] \mapsto \left[x_0 : \cdots : x_3 : -\sqrt[5]{x_0^5 + \cdots + x_3^5}\right]$$

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Non-Example 2: small complex structure limit

- Another quintic in $\mathbb{CP}^4$:
  1. [Kra09] $f_\pm = \left(x_0(x_1^2 + x_2^2 + x_3^2 - x_4^2) - (x_1^3 + x_2^3 + x_3^3 - \frac{1}{2}x_4^3) \pm \epsilon x_0^3\right)$
     smoothing of ordinary double point $(1:0:0:0:0)$
     Has $Z(f_+) \cong \mathbb{RP}^3$, $Z(f_-) \cong \mathbb{RP}^3 \# S^1 \times S^2$
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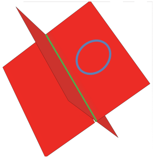

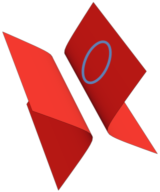

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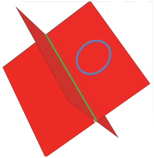

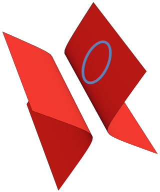

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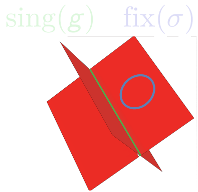

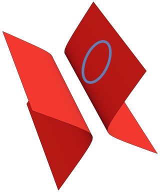

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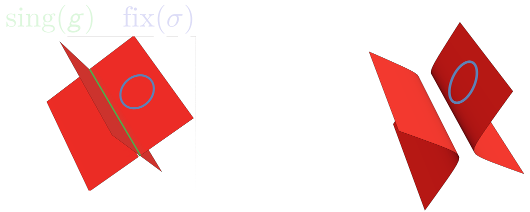

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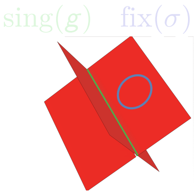

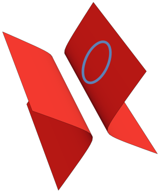

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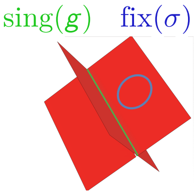

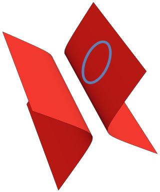

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Conjectural example 3: quadric intersect quartic

- Construction of quadric intersect quartic in $\mathbb{CP}^5$, also Calabi-Yau
  1. Circle $c_{aff} = x_1^2 + x_2^2 - 1$, quartic $q_{aff} = x_3^4 + x_4^4 + x_5^4 - 1$
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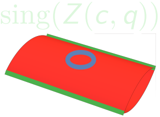

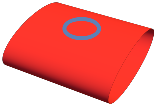

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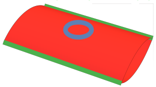

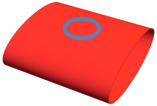

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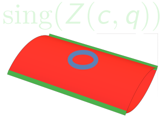

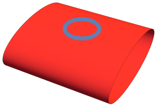

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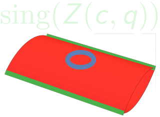

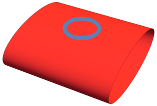

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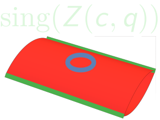

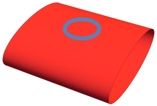

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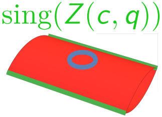

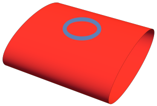

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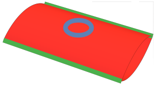

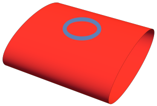

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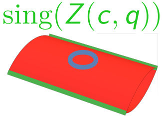

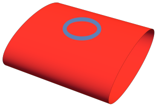

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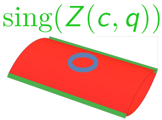

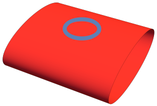

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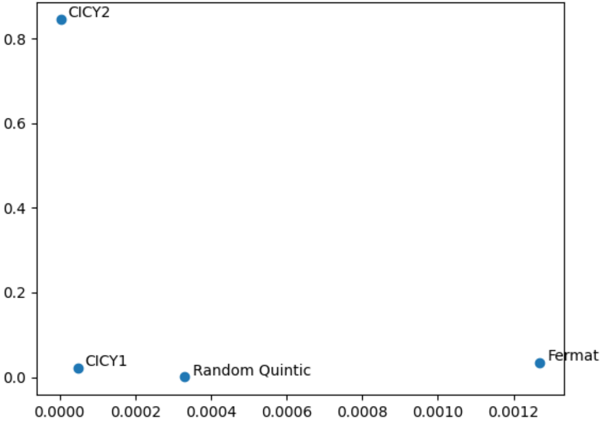
Experimental results: 1-forms and their zeros

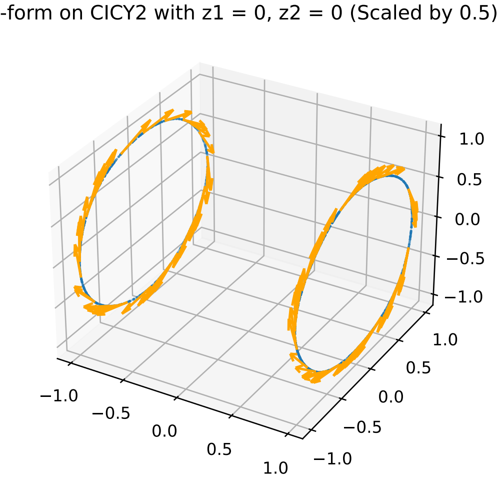
1. **Fermat:** non-example 1; no harmonic 1-form.
2. **Random Quintic:** non-example 2; harmonic 1-form must have zeros
3. **CICY1:** large perturbation $\epsilon = \frac{1}{4}$, harmonic 1-form may have zeros
4. **CICY2:** small perturbation $\epsilon = \frac{1}{100}$, conjecture no zeros

y-axis: $\min |\lambda|$

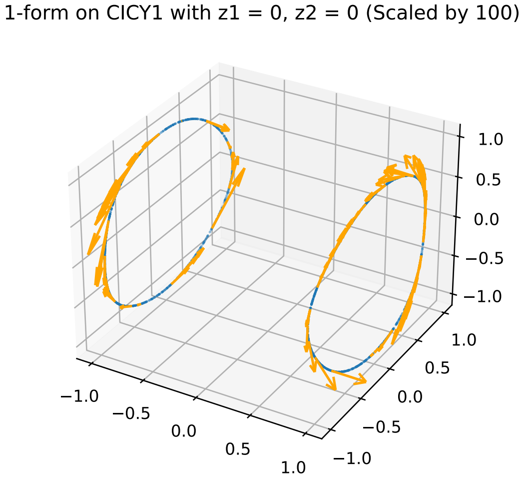
x-axis:
harmonic loss
$\frac{||\Delta\lambda||_{L^1}}{||\lambda||_{L^2}}$



Plots of approximately harmonic 1-forms



CICY2 (close to singular limit)



CICY1

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- Want: $\Delta_{\text{CY}}\lambda = 0$
- If $\omega_{\text{CY}} \approx \omega_{\text{approx}}$, get nearby $\tilde{\lambda}$ s.t. $\Delta_{\text{CY}}\tilde{\lambda} = 0$. Still $|\tilde{\lambda}| > 0$?
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Thank you for the attention!

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