

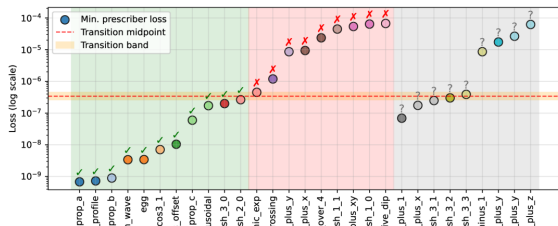
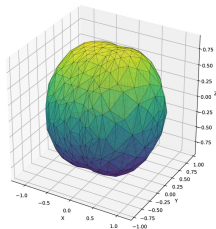
Daniel Platt: Non-uniqueness and symmetries for the Nirenberg problem using computer assistance

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The Nirenberg problem asks: which functions are the curvature of a metric on a sphere that is conformal to the round metric? This reduces to solving a non-linear partial differential equation (PDE) for a function on the sphere. In the recent arXiv:2602.12368, machine learning was used to conjecture that certain curvatures admit a solution to the PDE, based on numerics. In the talk, I will explain how to upgrade one of these conjectural examples to a proven example using a numerically verified proof. Beyond existence of a single solution, that method conveniently proves non-uniqueness and gives the exact symmetries of the solutions. This reports on results in arXiv:2603.29544.

Nirenberg problem

- ▶ Which $K : S^2 \rightarrow \mathbb{R}$ is curvature of a metric pointwise conformal to round metric?

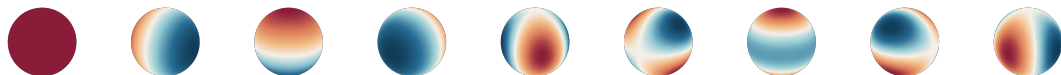


Images from "A Machine Learning Approach to the Nirenberg Problem"

- ▶ Equivalent: does $1 - \Delta u - Ke^{2u} = 0$ have solution $u : S^2 \rightarrow \mathbb{R}$?
- ▶ Unique? Symmetry? What does it look like?

Nirenberg results

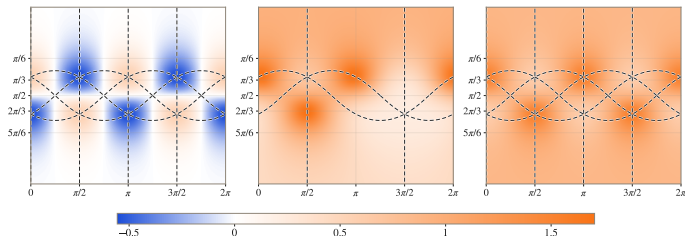
- ▶ Real spherical harmonics $Y_{l,m}$ for $l \geq 0$, $m \in \{-l, \dots, l\}$



- ▶ $Y_{3,2}$ has symmetry T_d (tetrahedral group) with $|T_d| = 24$

Theorem

For $K = Y_{3,2}$ there are at least **five solutions** to the equation $1 - \Delta u - Ke^{2u} = 0$. One has symmetry group T_d , four have symmetry group S_3 .



- ▶ Potential applications: min surfaces, ODEs, holonomy, Einstein metrics

Proof overview

- ▶ Solve equation $\mathcal{F}(u) := 1 - \Delta u - Ke^{2u} = 0$
- ▶ Function spaces: for $u : S^2 \rightarrow \mathbb{R}$

$$\|u\|_{L^2} := \left(\int_{S^2} u^2 \right)^{1/2} \quad \text{“}L^2\text{-norm”}$$

$$\|u\|_{H^2} := \|u\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla \nabla u\|_{L^2} \quad \text{“Sobolev norm”}$$

$$L^2 = \{u : S^2 \rightarrow \mathbb{R} : \|u\|_{L^2} < \infty\}, \quad H^2 = \{u : S^2 \rightarrow \mathbb{R} : \|u\|_{H^2} < \infty\}$$

- ▶ Approximate solution $u_0 \in H^2$, i.e. $\mathcal{E} := \mathcal{F}(u_0)$ small
- ▶ Linearisation $\mathcal{L}(u) := d\mathcal{F}_{u_0}(u) = -\Delta u - 2Ke^{2u_0} \cdot u$, $\mathcal{L} : H^2 \rightarrow L^2$
- ▶ Higher order terms $\mathcal{N}(u) := -Ke^{2u_0}(e^{2u} - 1 - 2u)$ s.t. “ $\mathcal{F} = \mathcal{E} + \mathcal{L} + \mathcal{N}$ ”

Theorem (Newton-Kantorovich)

Let $r > 0$ such that for $u_1, u_2 \in B_r := \{u \in H^2 : \|u\| \leq r\}$ we have

$$\|\mathcal{E}\|_{L^2} \leq c_1, \|\mathcal{L}^{-1}\| \leq c_2, \|\mathcal{N}u_1 - \mathcal{N}u_2\|_{L^2} \leq c_3(\|u_1\|_{H^2} + \|u_2\|_{H^2})\|u_1 - u_2\|_{H^2}$$

and $c_2c_1 + c_2c_3r^2 \leq r$, $2c_2c_3r < 1$, then ex. $v \in B_r$ such that $\mathcal{F}(u_0 + v) = 0$.

Proof step 1: bound the residual \mathcal{E}

- ▶ $\mathcal{F}(u) := 1 - \Delta u - Ke^{2u} = 0$, $\mathcal{E} := \mathcal{F}(u_0)$

Theorem (Newton-Kantorovich)

Let $r > 0$ such that for $u_1, u_2 \in B_r := \{u \in H^2 : \|u\| \leq r\}$ we have

$$\|\mathcal{E}\|_{L^2} \leq c_1, \|\mathcal{L}^{-1}\| \leq c_2, \|\mathcal{N}u_1 - \mathcal{N}u_2\|_{L^2} \leq c_3(\|u_1\|_{H^2} + \|u_2\|_{H^2})\|u_1 - u_2\|_{H^2}$$

and $c_2c_1 + c_2c_3r^2 \leq r$, $2c_2c_3r < 1$, then ex. $v \in B_r$ such that $\mathcal{F}(u_0 + v) = 0$.

- ▶ Compute some $u_0 = \sum_{0 \leq l, |m| \leq 50} a_{lm} Y_{l,m}$ (e.g. neural network)
- ▶ Bound $\|u_0\|_\infty \leq \sum |a_{lm}| \cdot \|Y_{l,m}\|_\infty < 1.58$ not sharp (😞 really Lipschitz)
- ▶ For $p = 50$ let \exp_p degree p Taylor poly of \exp , then:
 $\|\exp(2u_0) - \exp_p(2u_0)\|_\infty \lesssim \frac{\|u_0\|_\infty}{(p+1)!}$ small
- ▶ $\|\mathcal{E}\|_{L^2} \leq \|1 - \Delta u_0 - K \exp_p(2u_0)\|_{L^2} + \|K(\exp(2u_0) - \exp_p(2u_0))\|_{L^2} < 2.2 \cdot 10^{-8}$

Where first summand computed in spherical harmonics coefficient space

Proof step 2: Bound the linearised operator

▶ $\mathcal{F}(u) := 1 - \Delta u - Ke^{2u} = 0$, $\mathcal{L}(u) = -\Delta u - 2Ke^{2u_0} \cdot u$, $\mathcal{L} : H^2 \rightarrow L^2$

▶ Want: explicit c s.t. $\|u\|_{H^2} \leq c \|\mathcal{L}u\|_{L^2}$

▶ Differential geometry (integrated Bochner formula):

$$\begin{aligned} \|u\|_{H^2} &\leq \sqrt{2} \|-\Delta u\|_{L^2} + \|u\|_{L^2} \leq \sqrt{2} \|-\Delta u - 2Ke^{2u_0} \cdot u\|_{L^2} + (1 + \|2Ke^{2u_0}\|_{\infty}) \|u\|_{L^2} \\ &\leq \sqrt{2} \|\mathcal{L}u\|_{L^2} + 37.8 \|u\|_{L^2} \end{aligned}$$

▶ $L = 33$ split $H^2 = \langle Y_{l,m} : \text{degree} < L \rangle \oplus \langle Y_{l,m} : \text{degree} \geq L \rangle \Rightarrow \mathcal{L} = \begin{pmatrix} A & C \\ C^* & D \end{pmatrix}$

▶ For A : $\|Au\|_{L^2} \geq 2.04 \|u\|_{L^2}$ fin.-dim. computation (🙄 tricky linear algebra)

▶ For D : $\|Du\|_{L^2} \geq \|\Delta u\|_{L^2} - \|2Ke^{2u_0} \cdot u\|_{L^2} \geq L^2 \|u\|_{L^2} - c \|u\|_{L^2} \geq 10^3 \|u\|_{L^2}$

▶ For C : $u \text{ deg} \geq L$, p projection onto $\text{deg} < L$, then:

$$\|Cu\|_{L^2} = \left\| - \underbrace{p\Delta u}_{\text{deg}(\Delta u) > L \Rightarrow = 0} - p(2Ke^{2u_0} u) \right\|_{L^2} \leq \|2Ke^{2u_0} u\|_{L^2} \leq 26.1 \|u\|_{L^2}$$

▶ $\Rightarrow \|\mathcal{L}u\|_{L^2} \geq 1.3 \cdot \|u\|_{L^2} \Rightarrow \|u\|_{H^2} \leq (\sqrt{2} + 1.3 \cdot 37.8) \|\mathcal{L}u\|_{L^2}$

Proof step 3: Higher order terms and conclusion

- $\mathcal{N}(u) := e^u - 1 - u$ (🙄 omitted some factors)

Theorem (Newton-Kantorovich)

Let $r > 0$ such that for $u_1, u_2 \in B_r := \{u \in H^2 : \|u\| \leq r\}$ we have

$$\|\mathcal{E}\|_{L^2} \leq c_1, \|\mathcal{L}^{-1}\| \leq c_2, \|\mathcal{N}u_1 - \mathcal{N}u_2\|_{L^2} \leq c_3(\|u_1\|_{H^2} + \|u_2\|_{H^2})\|u_1 - u_2\|_{H^2}$$

and $c_2c_1 + c_2c_3r^2 \leq r$, $2c_2c_3r < 1$, then ex. $v \in B_r$ such that $\mathcal{F}(u_0 + v) = 0$.

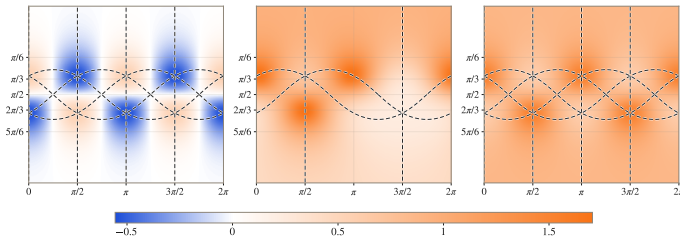
$$\begin{aligned} & \|\mathcal{N}u_1 - \mathcal{N}u_2\|_{L^2} \\ &= \|e^{u_1} - u_1 - (e^{u_2} - u_2)\|_{L^2} \\ &\leq e^{\max\{\|u_1\|_\infty, \|u_2\|_\infty\}} (\|u_1\|_\infty + \|u_2\|_\infty) \|u_1 - u_2\|_{L^2} \quad (\text{property of } e) \\ &\leq \underbrace{C_{\text{emb}} e^{\max\{\|u_1\|_{H^2}, \|u_2\|_{H^2}\}}}_{=: c_3} (\|u_1\|_{H^2} + \|u_2\|_{H^2}) \|u_1 - u_2\|_{L^2} \quad (\text{Sobolev embedding}) \end{aligned}$$

- Check $c_2c_1 + c_2c_3r^2 \leq r$, $2c_2c_3r < 1$ satisfied for $r = 3.1 \cdot 10^{-4} \rightsquigarrow \hat{u}$ exists

Symmetries

Theorem

For $K = Y_{3,2}$ there are at least **five solutions** to the equation $1 - \Delta u - Ke^{2u} = 0$. One has symmetry group T_d , four have symmetry group S_3 .



- ▶ Start with u_0 **exactly** T_d/S_3 -invariant
- ▶ Define L^2 and H^2 to be only **invariant** functions;
 \mathcal{L}, \mathcal{N} also invariant $\Rightarrow \hat{u}$ **invariant**
- ▶ For S_3 case rule out extra symmetries: $g \in T_d \setminus S_3$, find x s.t.
 $|u_0(x) - g^*(u_0)(x)| > 0.1$, then
 $|\hat{u}(x) - g^*(\hat{u})(x)| \geq |u_0(x) - g^*(u_0)(x)| - 2\|\hat{u} - u_0\|_\infty \geq 0.1 - 10^{-2} \neq 0$
 $\Rightarrow g^*(\hat{u}) \neq \hat{u}$ (😞 extra trick: symmetries at most T_d)

- ▶ **AI agent** can write the whole thing in one go (e.g. OpenAI Codex)

This repository accompanies the preprint <https://arxiv.org/abs/2603.29544> . It gives a numerically verified proof that for curvature prescriber $Y_{(3,2)}$ the Nirenberg problem has a solution.

Please re-do the proof for $Y_{(3,1)}$. You need to find a way to produce an approximate solution `u0.json` for this. This is not included in the repository. Write a solver using a pseudospectral method using `float64`. Do a non-rigorous `float64` test that the residual is of size 10^{-8} . Put all these new files into a top-level directory "solver". Once this is the case, run the numerically verified pipeline on this curvature prescriber using your generated `u0.json`. Skip the

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- ▶ Problem: completely **unreviewable**, **non-rigorous arithmetic** in several places
- ▶ Useful for **prototype/viability** (identify too slow computations)
- ▶ Give **code structure** (methods/classes): fewer non-rigour errors and easy to find
- ▶ Produce **unit tests** to find potential issues (needn't be trusted)

Thank you for the attention!