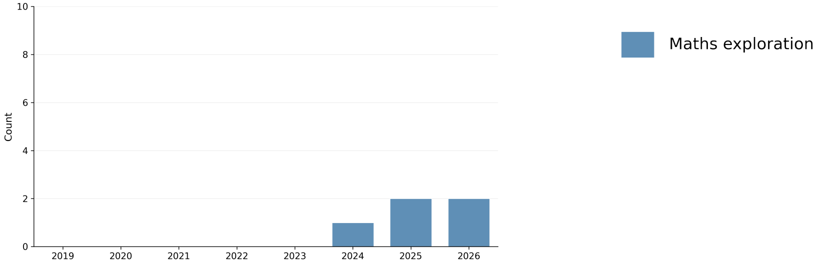
From Physics-informed Neural Networks to Numerically Verified Proofs in Geometric Analysis

Daniel Platt (Imperial College London)
Isaac Newton Institute, 21 Apr 2026

There are many elliptic equations in geometric analysis for which it is not known whether a solution exists. In many cases, good numerical solutions using PINN and other methods are known. There are only few numerically verified proofs in geometry, even though the equations are simple from an analysis point of view. In the talk I will mention the limited scenarios in which numerically verified proofs succeeded and I will point out the two places where geometry makes proving spectral gaps and interior estimates difficult, even for the simplest differential operators.

Exploration and numerically verified proofs



- Numerically verified proofs only in special cases
  I.e.: S^2 or symmetry reduction to $\mathbb{R}$ or $\mathbb{R}^2$
- Open questions with good numerics:
  Does a solution exist?
  (E.g. minimal surfaces [Sch21])
  Solution is known to exist, but what does it look like? (E.g. Calabi-Yau [DPQB25])
- Numerics for spectra of (twisted) Laplacian used in Physics

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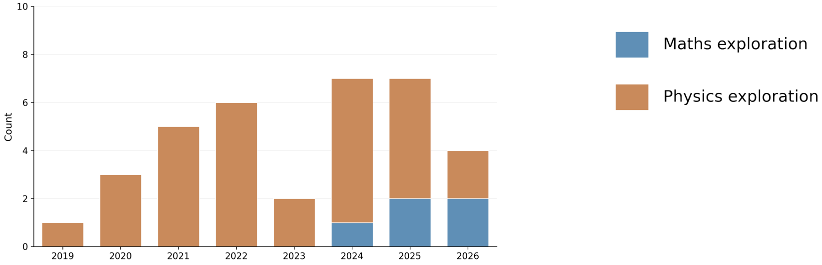

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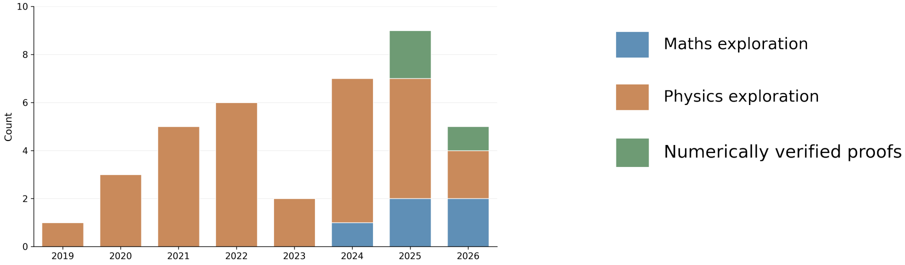

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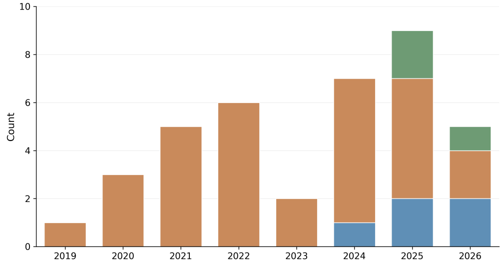

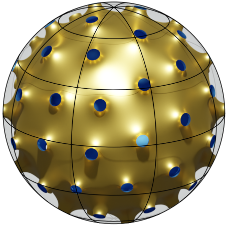

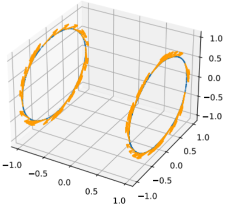

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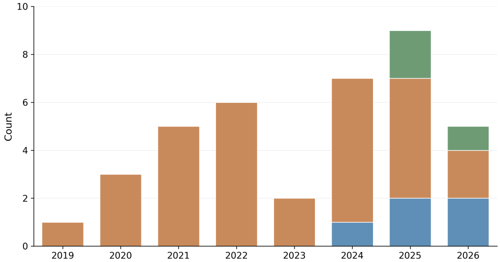

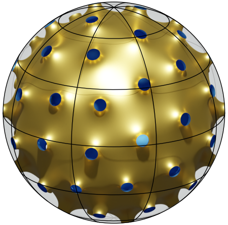

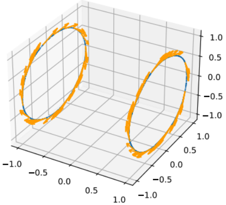

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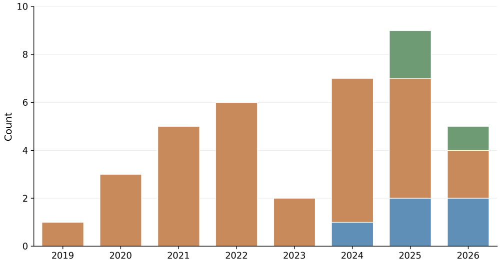

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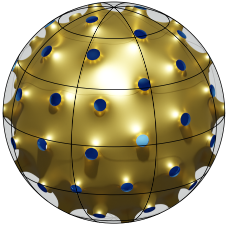
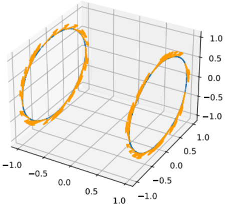
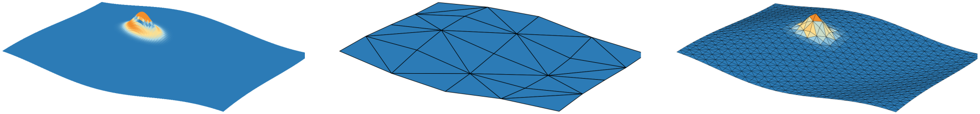




Difficulty 1: Spectral gap on manifolds

- General purpose theorems, e.g.:
  - Li-Yau $\lambda_1(\Delta) \geq \{\text{Ricci, diam}\}$
  - Cheeger $\lambda_1(\Delta) \geq \{\text{Cheeger const.}\}$
- On nice spaces use (pseudo)-spectral basis [Pla26, RT19], cf. J. Henkel's talk
- Symmetries $\rightsquigarrow$ ODE $\rightsquigarrow$ shooting problem [BH26, Wan25]
- Discretisation (FEM, DEC) $\Rightarrow \lambda_1^h$
- Difficult to compare $\lambda_1 \geq \{h, \lambda_1^h\}$ depending on geometry
  E.g. [HSZ24, ST20, Liu15]



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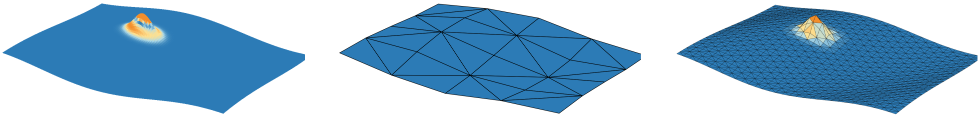

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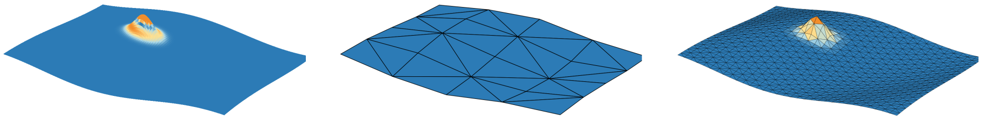

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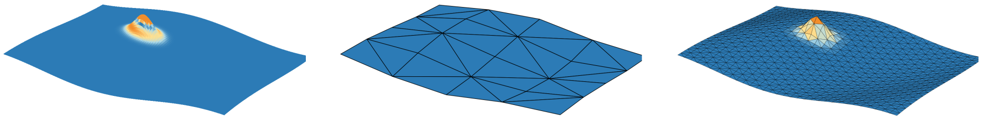

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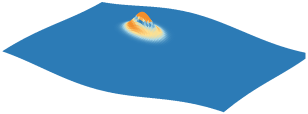
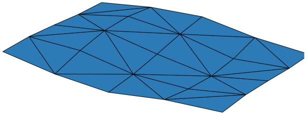
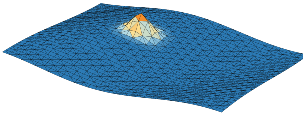

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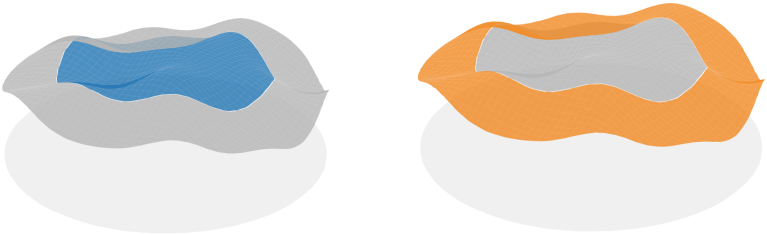
- On $\Omega \subset \mathbb{R}^2$ with Dirichlet: $\left\|\nabla^2 u\right\|_{L^2} = \|\Delta u\|_{L^2}$
- On surface with |curvature| $\leq K$: $\left\|\nabla^2 u\right\|_{L^2} \lesssim \|\Delta u\|_{L^2} + K\|u\|_{L^2}$
- $W^{k,p}$ and $C^{k,\alpha}$ dependence on curvature?
- Computational methods for these constants?



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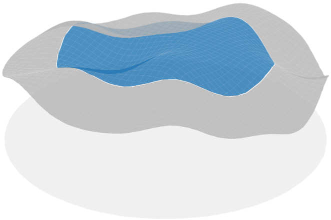
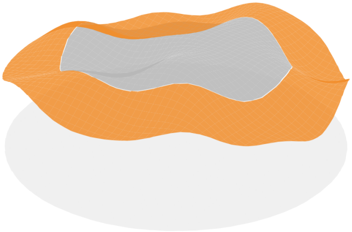

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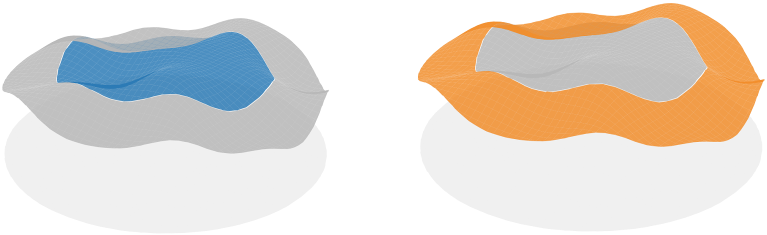

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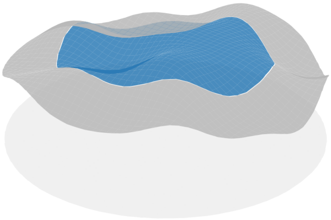

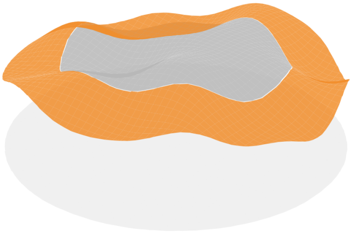

Thank you for the attention!

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