Numerically verified proofs aided by machine learning

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Erlangen Hub, 29 May 2026

Many methods are available to approximately solve all sorts of equations: ODEs, PDEs, polynomial systems, algebraic equations. Through recent advances it has become possible to turn these approximate solutions into rigorous existence proofs in some cases. I will explain how numerically verified proofs work and present several examples of them: (1) to get started, the classical problem of root enclosures for polynomials in one variable; (2) the main example is the Nirenberg problem, where in arXiv:2602.12368 a PDE was solved using a PINN, and in arXiv:2603.29544 I proved that there exists a genuine solution to the PDE; (3) another small application that's to be decided, possibly related to topological data analysis.

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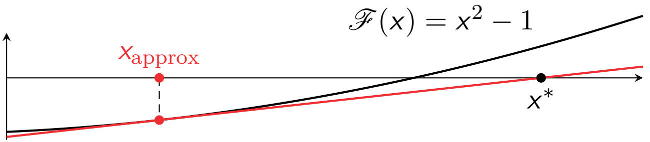
Example 1: Baby example

- Zero of $\mathscr{F}(x) = x^2 - 1$. Start with x_{approx}. Newton iteration: does it converge?



- Computer: $x_{approx} = 0.999$. Write $x = x_{approx} + v$. Taylor:

$$\mathscr{F}(x) = \mathscr{F}(x_{approx}) + \mathscr{L}(v) + \mathscr{N}(v), \quad \mathscr{L} \text{ linear}, \quad \mathscr{N} \text{ h.o.t.}$$

- Error: $|\mathscr{E}| := |\mathscr{F}(x_{approx})| \leq 0.002$

Theorem (Newton-Kantorovich)

Let $r > 0$ such that for $u_1, u_2 \in B_r := \{u \in \mathbb{R} : |u| \leq r\}$ we have

$$|\mathscr{E}| \leq c_1, ||\mathscr{L}^{-1}|| \leq c_2, |\mathscr{N} u_1 - \mathscr{N} u_2| \leq c_3(|u_1| + |u_2|)|u_1 - u_2|$$

and $c_2 c_1 + c_2 c_3 r^2 \leq r$, $2 c_2 c_3 r < 1$, then ex. $v \in B_r$ such that $\mathscr{F}(u_0 + v) = 0$.

- For $x_{\text{approx}} = 0.999$: $c_1 = 0.002$, $c_2 = 0.501$, $c_3 = 1$ ✅ for $r = 0.002$

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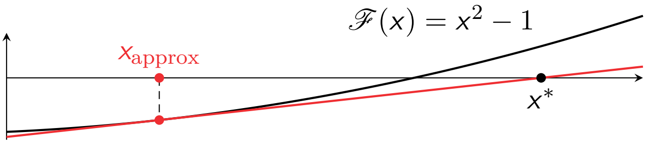

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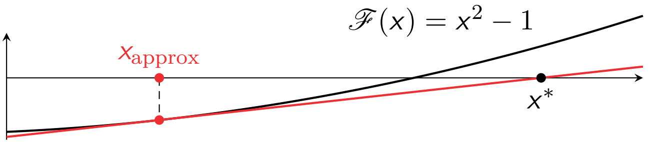

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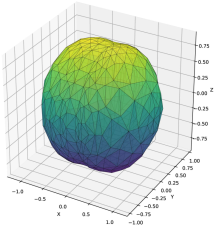
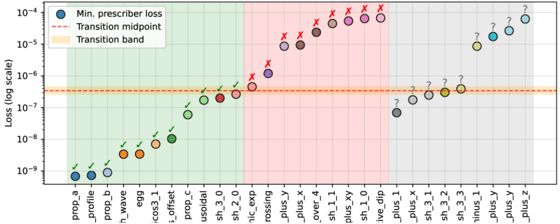
- Which $K : S^2 \to \mathbb{R}$ is curvature of a metric pointwise conformal to round metric?





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- Equivalent: does $1 - \Delta u - Ke^{2u} = 0$ have solution $u : S^2 \to \mathbb{R}$?
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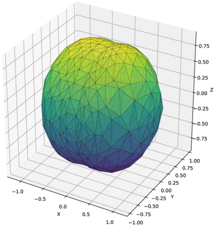

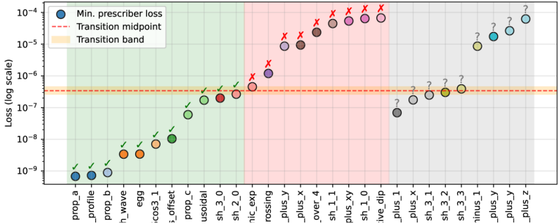

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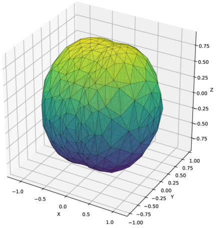

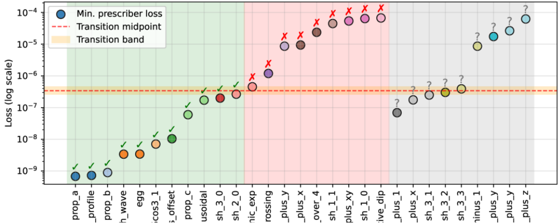

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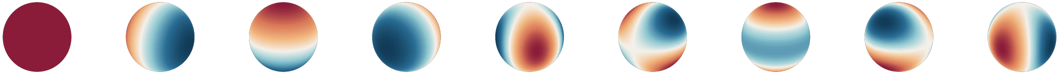
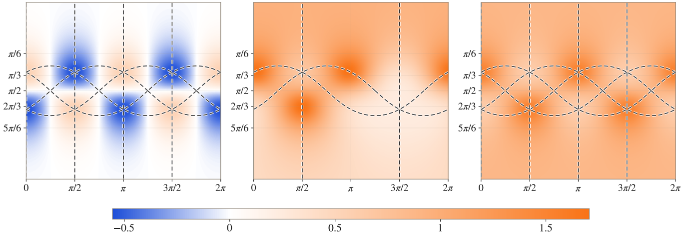


- $Y_{3,2}$ has symmetry T_d (tetrahedral group) with $|T_d| = 24$

Theorem

For $K = Y_{3,2}$ there are at least five solutions to the equation $1 - \Delta u - Ke^{2u} = 0$. One has symmetry group T_d, four have symmetry group S_3.



- Potential applications: min surfaces, ODEs, bottlenecks, equilibria

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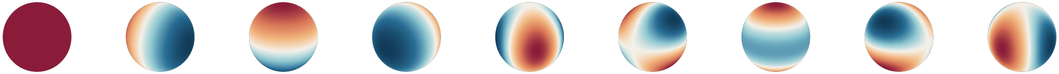

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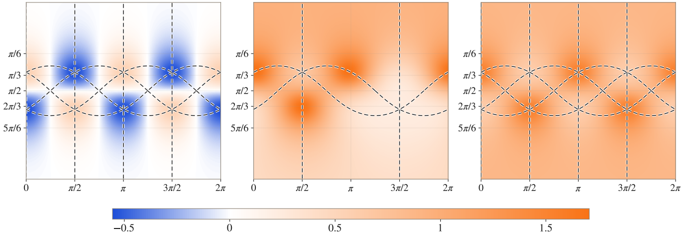

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- Approximate solution $u_0 \in H^2$, i.e. $\mathscr{E} := \mathscr{F}(u_0)$ small
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- Compute some $u_0 = \sum_{0 \leq l, |m| \leq 50} a_{lm} Y_{l,m}$ (e.g. neural network)
- Bound $||u_0||_\infty \leq \sum |a_{lm}| \cdot ||Y_{l,m}||_\infty < 1.58$ not sharp (🤥 really Lipschitz)
- For $p = 50$ let $\exp_p$ degree p Taylor poly of exp, then: $\left|\left|\exp(2u_0) - \exp_p(2u_0)\right|\right|_\infty \lesssim \frac{||u_0||_\infty}{(p+1)!}$ small
- $||\mathscr{E}||_{L^2} \leq \left|\left|1 - \Delta u_0 - K\exp_p(2u_0)\right|\right|_{L^2} + \left|\left|K(\exp(2u_0) - \exp_p(2u_0))\right|\right|_{L^2} < 2.2 \cdot 10^{-8}$
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$$||\mathscr{E}||_{L^2} \leq c_1, ||\mathscr{L}^{-1}|| \leq c_2, ||\mathscr{N} u_1 - \mathscr{N} u_2||_{L^2} \leq c_3(||u_1||_{H^2} + ||u_2||_{H^2})\,||u_1 - u_2||_{H^2}$$

and $c_2 c_1 + c_2 c_3 r^2 \leq r$, $2c_2 c_3 r < 1$, *then ex.* $v \in B_r$ *such that* $\mathscr{F}(u_0 + v) = 0$.

- Compute some $u_0 = \sum_{0 \leq l, |m| \leq 50} a_{lm} Y_{l,m}$ (e.g. neural network)
- Bound $||u_0||_\infty \leq \sum |a_{lm}| \cdot ||Y_{l,m}||_\infty < 1.58$ not sharp (🤥 really Lipschitz)
- For $p = 50$ let $\exp_p$ degree p Taylor poly of exp, then:
  $||\exp(2u_0) - \exp_p(2u_0)||_\infty \lesssim \frac{||u_0||_\infty}{(p+1)!}$ small
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  Where first summand computed in spherical harmonics coefficient space

Ex 2: Proof step 1: bound the residual $\mathscr{E}$

- $\mathscr{F}(u) := 1 - \Delta u - Ke^{2u} = 0$, $\mathscr{E} := \mathscr{F}(u_0)$

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$$\mathscr{L} = \begin{pmatrix} A & C \\ C^* & D \end{pmatrix}$$

- A : $||Au||_{L^2} \geq 2.04\,||u||_{L^2}$ fin.-dim. computation (🤥 tricky linear algebra)
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Ex 2: Proof step 3: Higher order terms and conclusion

- $\mathcal{N}(u) := u^2$ (🤥 omitted some terms)

Theorem (Newton-Kantorovich)

Let $r > 0$ such that for $u_1, u_2 \in B_r := \{u \in H^2 : ||u|| \le r\}$ we have

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- Check $c_2c_1 + c_2c_3r^2 \le r$, $2c_2c_3r < 1$ satisfied for $r = 3.1 \cdot 10^{-4}$ $\rightsquigarrow$ $\hat{u}$ exists

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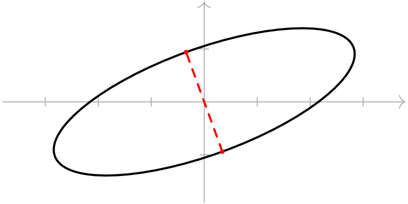
Example 3: bottlenecks of submanifolds

- $\gamma : I \to \mathbb{R}^2$ curve
- $p = \gamma(t_1), q = \gamma(t_2)$ bottleneck if critical point for distance function



- E.g. $\gamma(t) = \begin{pmatrix} 3\cos t \cos 20° - \sin t \sin 20° \\ 3\cos t \sin 20° + \sin t \cos 20° \end{pmatrix}$
- $D(s,t) = \frac{1}{2}|\gamma(t) - \gamma(s)|^2_{\text{Eucl}}$, $\qquad \mathscr{F}(s,t) = \nabla D(s,t)$
- $x_{\text{approx}} = (s_0, t_0) = (1.571,\ 4.712)$, $\mathscr{E} = \mathscr{F}(x_{\text{approx}}) \rightsquigarrow |\mathscr{E}|_\infty \leq 2.1 \cdot 10^{-3}$
- $\mathscr{L} = d\mathscr{F}_{x_{\text{approx}}} = \text{Hess}\, D(x_{\text{approx}})$, $|\mathscr{L}^{-1}|_\infty \leq 0.501$
- $|\mathscr{N}(v_1) - \mathscr{N}(v_2)|_\infty \leq 56\,(|v_1|_\infty + |v_2|_\infty)\,|v_1 - v_2|_\infty$
- Thus $c_1 = 2.1 \cdot 10^{-3}$, $c_2 = 0.501$, $c_3 = 56$. For $r = 1.1 \cdot 10^{-3}$ have $c_2 c_1 + c_2 c_3 r^2 < r$, $\qquad 2 c_2 c_3 r < 1 \rightsquigarrow$ bottleneck exists near x_{approx}

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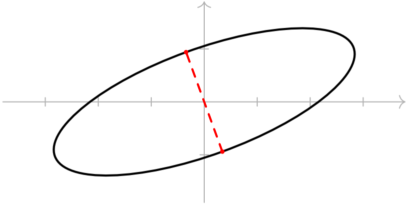

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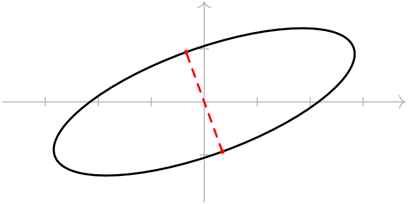

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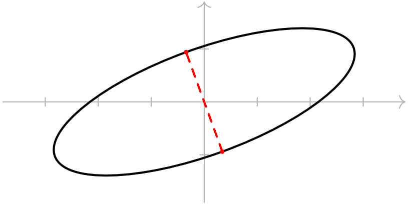

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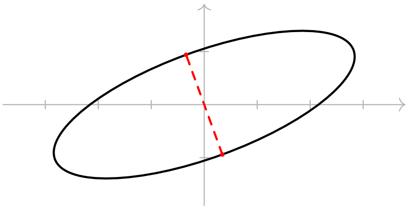

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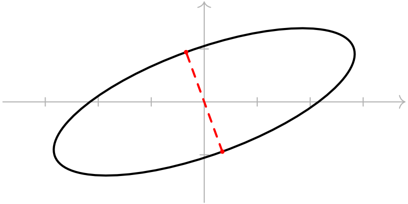

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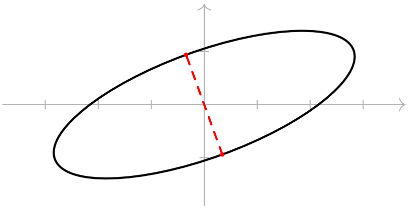

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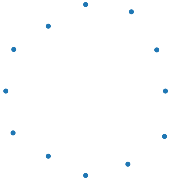
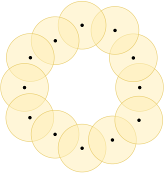
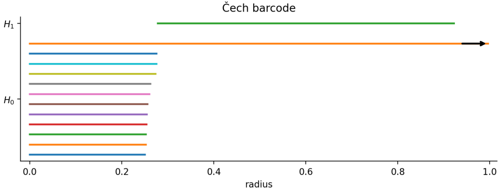
Example 5: point clouds for given barcodes

- Persistent homology: point cloud $\to$ barcode







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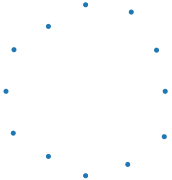

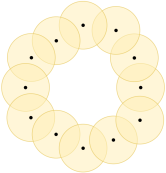

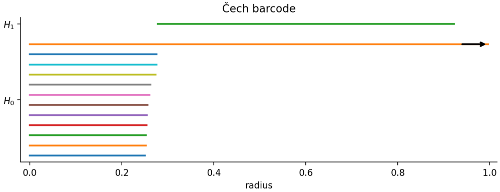

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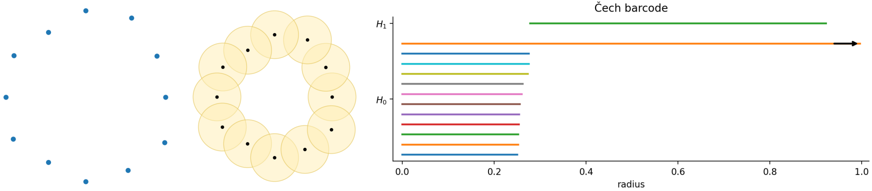

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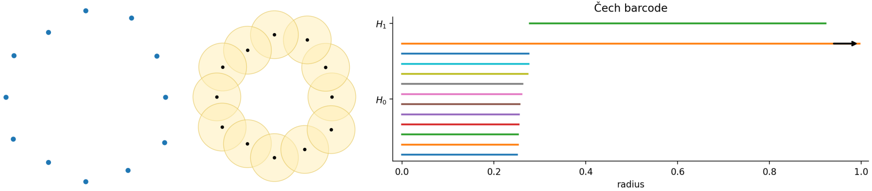

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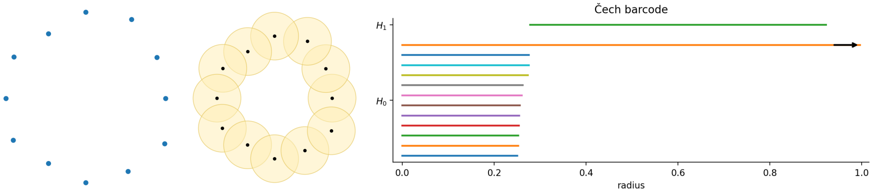

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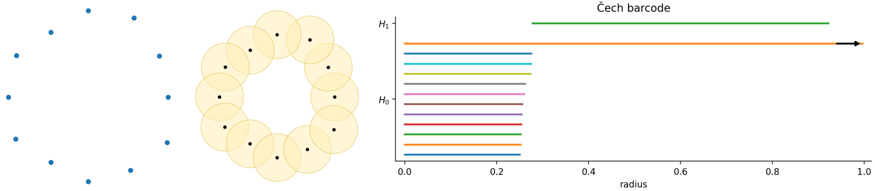
- "Gauge fixing", i.e. represent $X_{\text{approx}} \in \mathbb{R}^{3\ell - 6}$, then $\mathscr{L}$ right-invertible $\rightsquigarrow$ works

Example 5: point clouds for given barcodes

- Persistent homology: point cloud $\to$ barcode



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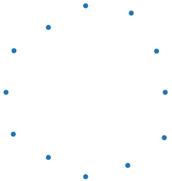

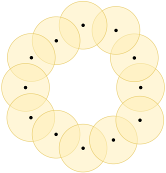

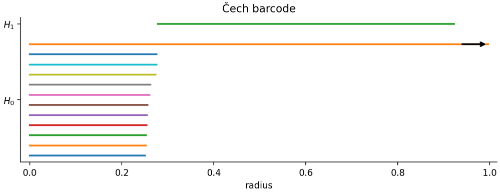

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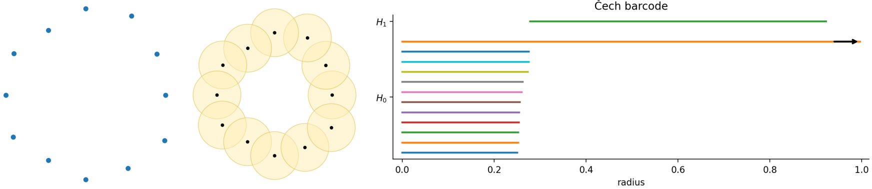

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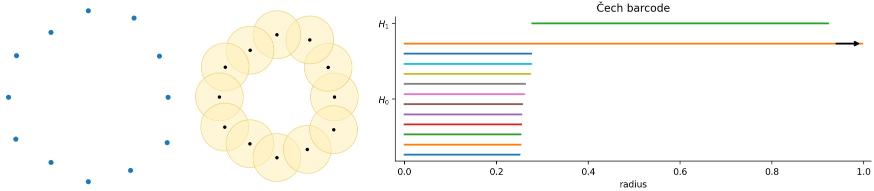

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Thank you for the attention!

References I

David Beers, Heather A Harrington, Jacob Leygonie, Uzu Lim, and Louis Theran.
Fibers of point cloud persistence.
arXiv preprint arXiv:2411.08201, 2024.

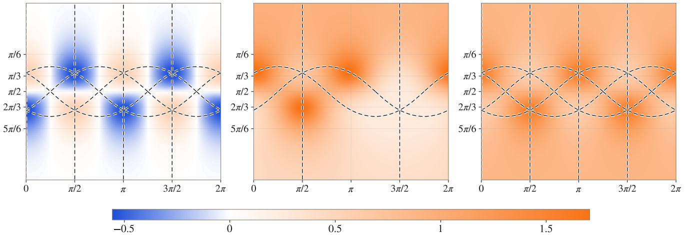
Iolo Jones.
Diffusion geometry, 2024.

Steve Oudot and Elchanan Solomon.
Inverse problems in topological persistence.
In *Topological Data Analysis: The Abel Symposium 2018*, pages 405–433.
Springer, 2020.

Symmetries

Theorem

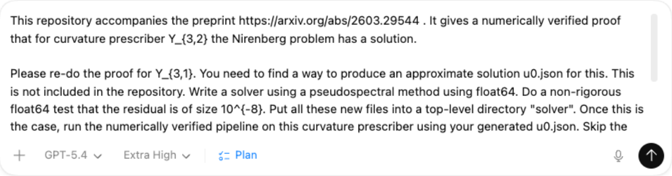
For $K = Y_{3,2}$ there are at least five solutions to the equation $1 - \Delta u - Ke^{2u} = 0$. One has symmetry group T_d, four have symmetry group S_3.



- Start with u_0 exactly T_d/S_3-invariant
- Define L^2 and H^2 to be only invariant functions;
  $\mathscr{L}, \mathscr{N}$ also invariant $\Rightarrow \hat{u}$ invariant
- For S_3 case rule out extra symmetries: $g \in T_d \setminus S_3$, find x s.t.
  $|u_0(x) - g^*(u_0)(x)| > 0.1$, then
  $|\hat{u}(x) - g^*(\hat{u})(x)| \geq |u_0(x) - g^*(u_0)(x)| - 2\,||\hat{u} - u_0||_\infty \geq 0.1 - 10^{-2} \neq 0$
  $\Rightarrow g^*(\hat{u}) \neq \hat{u}$ (🤥 extra trick: symmetries at most T_d)

AI coding tool usage

- AI agent can write the whole thing in one go (e.g. OpenAI Codex)



- Problem: completely unreviewable, non-rigorous arithmetic in several places
- Useful for prototype/viability (identify too slow computations)
- Give code structure (methods/classes): fewer non-rigour errors and easy to find
- Produce unit tests to find potential issues (needn't be trusted)